Numerical Weather Prediction

An improved treatment of the coriolis terms in the Perturbation Forecast (PF) model

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James Beck and Andy White

email: nwp_publications@metoffice.com

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Abstract

The Met Office four-dimensional variational assimilation system (4D-Var) uses a Perturbation Forecast (PF) model to predict the short-term effect on the forecast of a small change in the initial conditions. Effectively, the PF model is simulating the difference between two forecast runs with slightly differing initial conditions. The accuracy of this simulation can be assessed by comparing the increment produced by the PF model at some later time with the difference between two nonlinear model forecasts, with the initial perturbation in the PF model being equal to the difference between the nonlinear forecasts at the initial time. Results from these experiments showed that the PF model increment was becoming smaller with time relative to the nonlinear 'increment'. These tests are known as linearisation tests. The reason for this damping has been traced to the treatment of the coriolis terms in the horizontal momentum equations of the PF model. A new treatment for the terms has been included in the PF model and is presented here, with results showing the removal of the damping and improved accuracy in the PF model results. The results obtained are now very similar to those gained when treating the coriolis terms in the manner they are treated in the New Dynamics. The stability of the scheme when applied to inertial oscillations and the linearised shallow water equations is examined.

1 Introduction

The coriolis terms in the horizontal momentum equations are treated implicitly in the New Dynamics [1]. This has the effect of coupling the u and v momentum equations in the Helmholtz equation and can lead to extensive stencils in the solution procedure. Some substitution of the equations can be performed prior to taking finite differences, which alleviates the problem. The terms can be treated in the same manner within the PF model, but this couples each grid point to every other in the adjoint, and is thus unworkable if an exact adjoint is to be constructed. The PF model must then deviate from the New Dynamics in its treatment of these terms for the sake of the ability to derive its adjoint.

2 Previous Treatment

The solution to this problem as previously implemented in the PF model amounts to a severe simplification of the scheme of Gadd [2]. The horizontal momentum equations are first written with only the coriolis terms on the right hand side, as

$$u'^{n+1}_a = u'^{n}_d + (1-\alpha_5)\Delta t f v'^{n}_d + \alpha_5 \Delta t f v'^{n+1}_a, \quad (1)$$

$$v'^{n+1}_a = v'^{n}_d - (1-\alpha_5)\Delta t f u'^{n}_d - \alpha_5 \Delta t f u'^{n+1}_a, \quad (2)$$

From (1) and (2),

$$u'^{n+1}_a = W_1 u'^n_d + W_2 v'^n_d, \quad (3)$$
$$v'^{n+1}_a = W_1 v'^n_d - W_2 u'^n_d, \quad (4)$$

where

$$W_1 = \frac{1-\alpha_5(1-\alpha_5)\beta^2}{1+\alpha_5^2\beta^2}, \quad (5)$$
$$W_2 = \frac{\beta}{1+\alpha_5^2\beta^2}, \quad (6)$$

with $\beta = f\Delta t$.

Having thus found the arrival point increments due to the coriolis terms acting alone, the horizontal momentum equations are then used to add on the contributions of the non-coriolis terms. The value of α_5 used in (5) and (6) is 0.6 to match the implicit weighting used for the coriolis terms in the new dynamics. At this point it should be noted that W_1 and W_2 both decrease monotonically as α_5 increases in the interval [0,1]. This means that the contribution of the coriolis terms decreases as α_5 is increased, and thus enhancement of the stability of the scheme results in a damping of the coriolis terms. This has the effect, through the Helmholtz solver, of causing the entire scheme to be damped as the coriolis terms are large terms in the horizontal momentum equations. The error on each timestep is only of the order of 1%, but as the terms are strictly damped this can never be recovered and the effect amplifies with time.

Results from linearisation tests using this scheme are shown in Figures 1 and 2, for the zonal wind and the pressure respectively. As will be the case in all linearisation test figures, the 'nonlinear perturbation' is obtained from the difference between two forecasts valid at the same time, the difference between which at the initial time is used as the initial perturbation in the PF model. At this initial time, the two nonlinear model runs are a 24 hour forecast (used as the linearisation state) and a 48 hour forecast. 12 hours after the initial time, the PF perturbation fields can clearly be seen to be damped in comparison with the nonlinear perturbations. The damping is also noticeably worse at high latitudes than in the tropics. Level 21 has been displayed as the most visible damping is seen at about this level.

Figure 3 shows the damping coefficent for all the prognostic variables at all levels after 12 hours. This coefficient is defined as the global root-mean-square of the perturbation predicted by the PF model divided by the global root-mean-square of the nonlinear perturbation. Good PF model performance is shown by coefficients of approximately unity. The figure shows damping coefficients below 0.7 for pressure, and below 0.8 for four of the prognostic variables, clearly demonstrating that the PF model is producing a damped solution.

3 The New Predictor-Corrector scheme

A new predictor-corrector scheme for the horizontal momentum equations has been developed and tested, and has yielded much improved results with the damping removed. The scheme proceeds as follows. A target discretisation for the horizontal momentum equations is

$$\frac{u'^{n+1} - u'^n_d}{\Delta t} = \alpha_3 \left[f_3 v' - \frac{c_{pd}}{r cos\phi}\left(\theta\delta_\lambda \Pi' - \theta\delta_r \Pi' \delta_\lambda r\right) - \frac{c_{pd}}{r cos\phi}\left(\theta'\delta_\lambda \Pi - \theta'\delta_r \Pi \delta_\lambda r\right)\right]^{n+1}$$

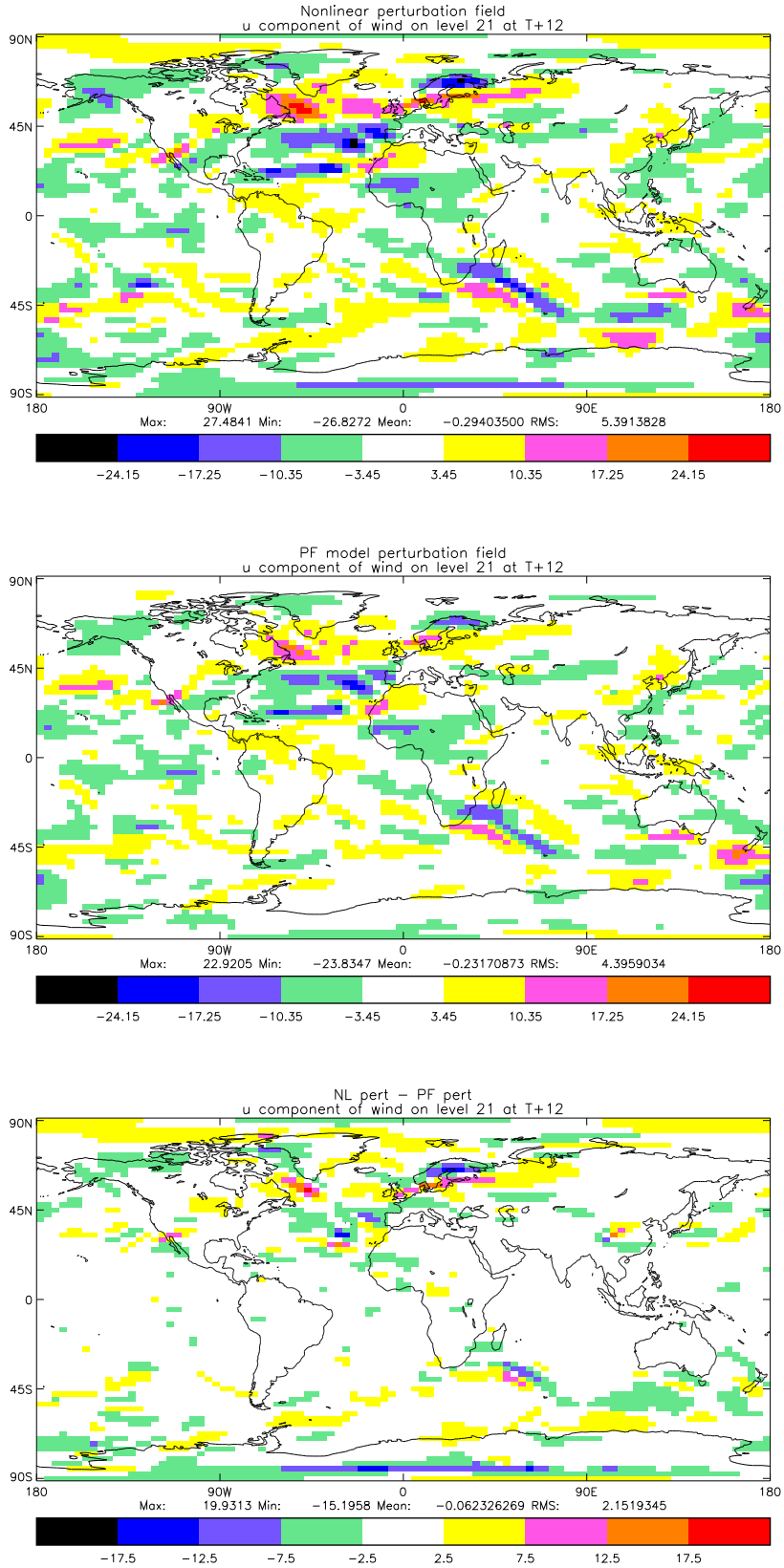

Figure 1: Zonal velocity perturbations from the Gadd scheme

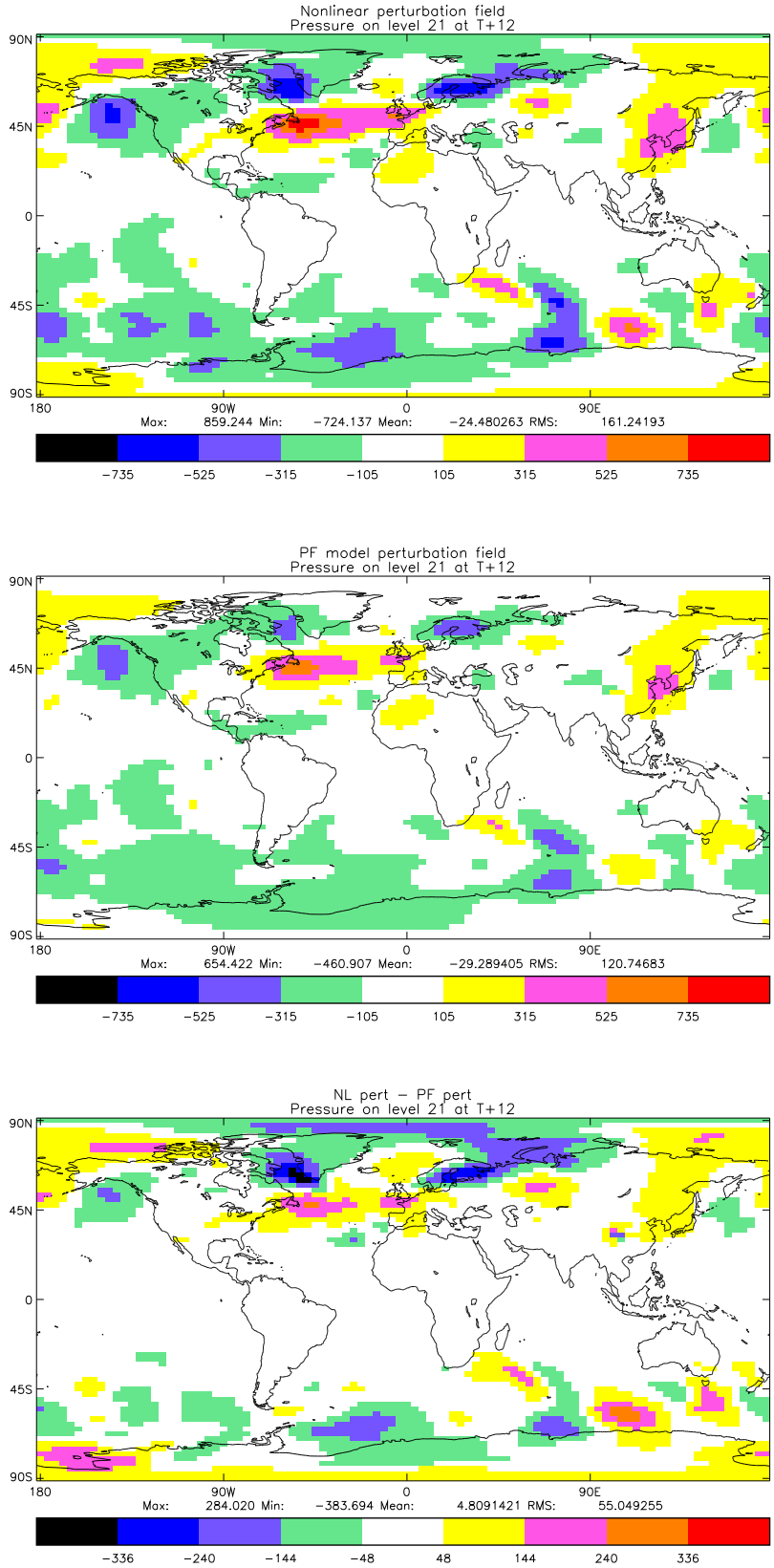

Figure 2: Pressure perturbations from the Gadd scheme

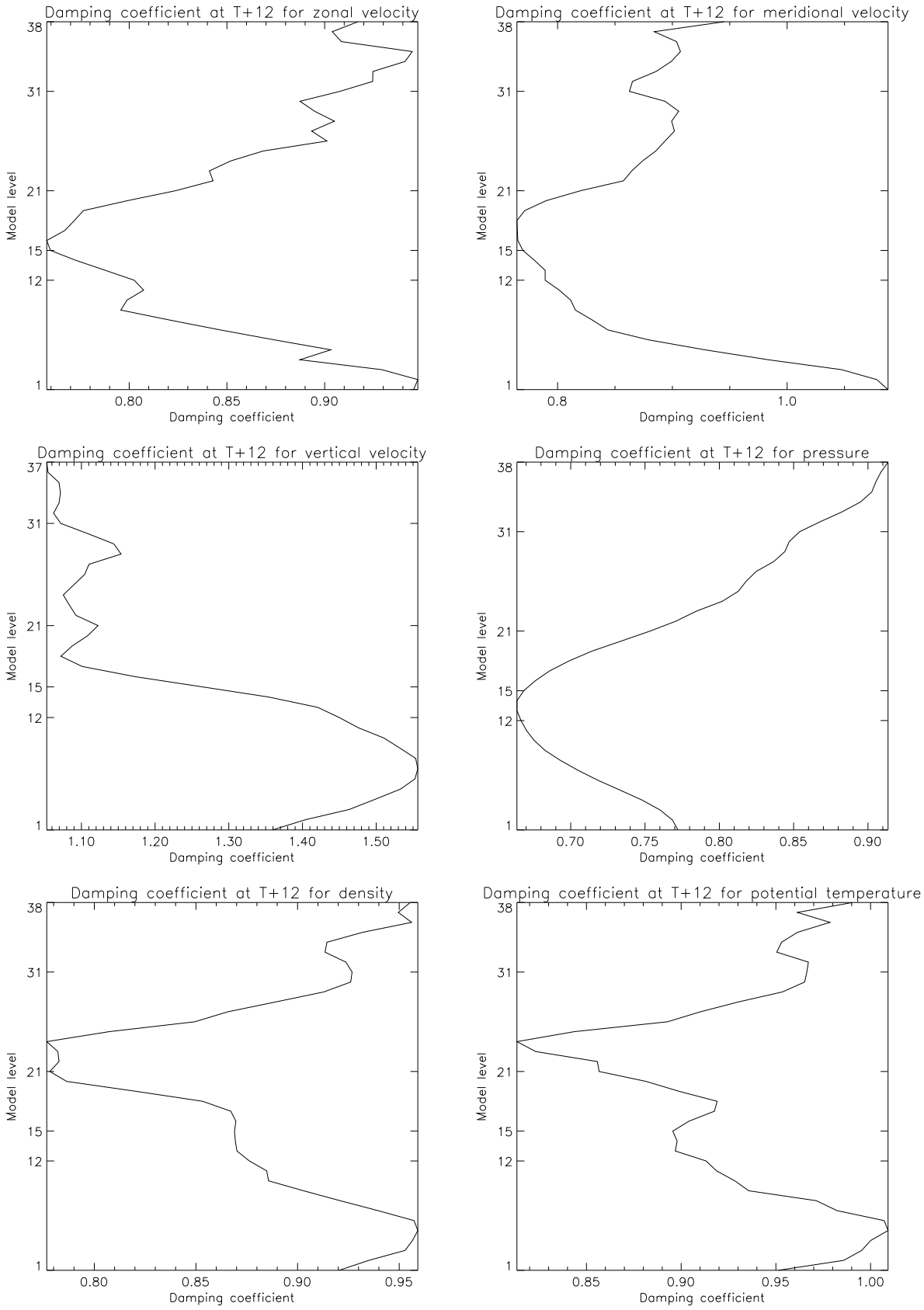

Figure 3: Damping coefficients from the Gadd scheme

$$+ (1-\alpha_3)\left[f_3 v' - \frac{c_{pd}}{r cos\phi}\left(\theta\delta_\lambda \Pi' - \theta\delta_r \Pi' \delta_\lambda r\right) - \frac{c_{pd}}{r cos\phi}\left(\theta'\delta_\lambda \Pi - \theta'\delta_r \Pi \delta_\lambda r\right)\right]_d^n$$
$$+ \left[\left(\underline{u'}.\underline{\nabla}\right) u\right]^n - \left(\frac{u' v tan\phi}{r}\right)^n \quad (7)$$

ignoring spatial interpolation issues. The first predictor step is then to calculate $u'^{(1)}$ explicitly, which is given by

$$u'^{(1)} = u'^n_d + \alpha_3 \Delta t\left[f_3 v'^n - \frac{c_{pd}}{r cos\phi}\left(\theta\delta_\lambda \Pi'^n - \theta\delta_r \Pi'^n \delta_\lambda r\right) - \frac{c_{pd}}{r cos\phi}\left(\theta'^*\delta_\lambda \Pi - \theta'^*\delta_r \Pi \delta_\lambda r\right)\right]$$
$$+ (1-\alpha_3)\,\Delta t\left[f_3 v' - \frac{c_{pd}}{r cos\phi}\left(\theta\delta_\lambda \Pi' - \theta\delta_r \Pi' \delta_\lambda r\right) - \frac{c_{pd}}{r cos\phi}\left(\theta'\delta_\lambda \Pi - \theta'\delta_r \Pi \delta_\lambda r\right)\right]_d^n$$
$$+\Delta t\left[\left(\underline{u'}.\underline{\nabla}\right) u\right]^n - \Delta t\left(\frac{u' v tan\phi}{r}\right)^n \quad (8)$$

in which the terms intended for implicit treatment are evaluated at time level n, except that the latest predictor θ'^* is used for the potential temperature perturbation. The u' equation is presented here and the v' equation is of a similar form and provides a predictor, $v'^{(1)}$,for the next time level meridional velocity component.

The first corrector applied to these equations provides second predictors $u'^{(2)}$ and $v'^{(2)}$ for the velocity values at the new time level. These correctors are

$$u'^{(2)} = u'^{(1)} + \alpha_3 \Delta t f_3\left(v'^{(1)} - v'^n\right) \quad (9)$$
$$v'^{(2)} = v'^{(1)} - \alpha_3 \Delta t f_3\left(u'^{(1)} - u'^n\right) \quad (10)$$

This has the effect of using the first predictor for u'^{n+1} and v'^{n+1} in (8) in place of the time level n value.

The second and final corrector provides the final value for the horizontal velocities at the next time level and is defined as

$$u'^{n+1} = u'^{(2)} - \alpha_3 \Delta t \frac{c_{pd}}{r cos\phi}\left(\theta\delta_\lambda \Pi'' - \theta\delta_r \Pi''\right) \quad (11)$$
$$v'^{n+1} = v'^{(2)} - \alpha_3 \Delta t \frac{c_{pd}}{r}\left(\theta\delta_\phi \Pi'' - \theta\delta_r \Pi''\right) \quad (12)$$

where $\Pi'' = \Pi^{n+1} - \Pi^n$. This has the effect of replacing the time level n values of the exner pressure perturbation with those from the next time level in (8). This couples the equations implicitly to the other prognostic equation and leads to a Helmholtz problem for the exner pressure tendency Π''. This final corrector is identical to the one currently employed, and so this scheme does not require any changes within the Helmholtz equation derivation and solution.

The results produced using this scheme can be seen in Figures 4 and 5, which are again on Level 21 after 12 hours. The damping relative to the nonlinear perturbation is evidently much reduced. This is confirmed by Figure 6, which shows the damping coefficient after 12 hours. The damping previously seen in the pressure, density and horizontal velocity fields is no longer evident. What is now apparent is that the PF model is amplifying the horizontal velocity fields near the surface, and the vertical velocity field throughout the boundary layer. This is due to an instability in the PF model equations unrelated to the coriolis terms which is beyond the scope of this report.

In order to assess the performance of this scheme relative to the New Dynamics implicit scheme, the New Dynamics scheme was coded into the PF model (recall that the reason this discretisation was not used in the first place was due to the adjoint, and not any issue

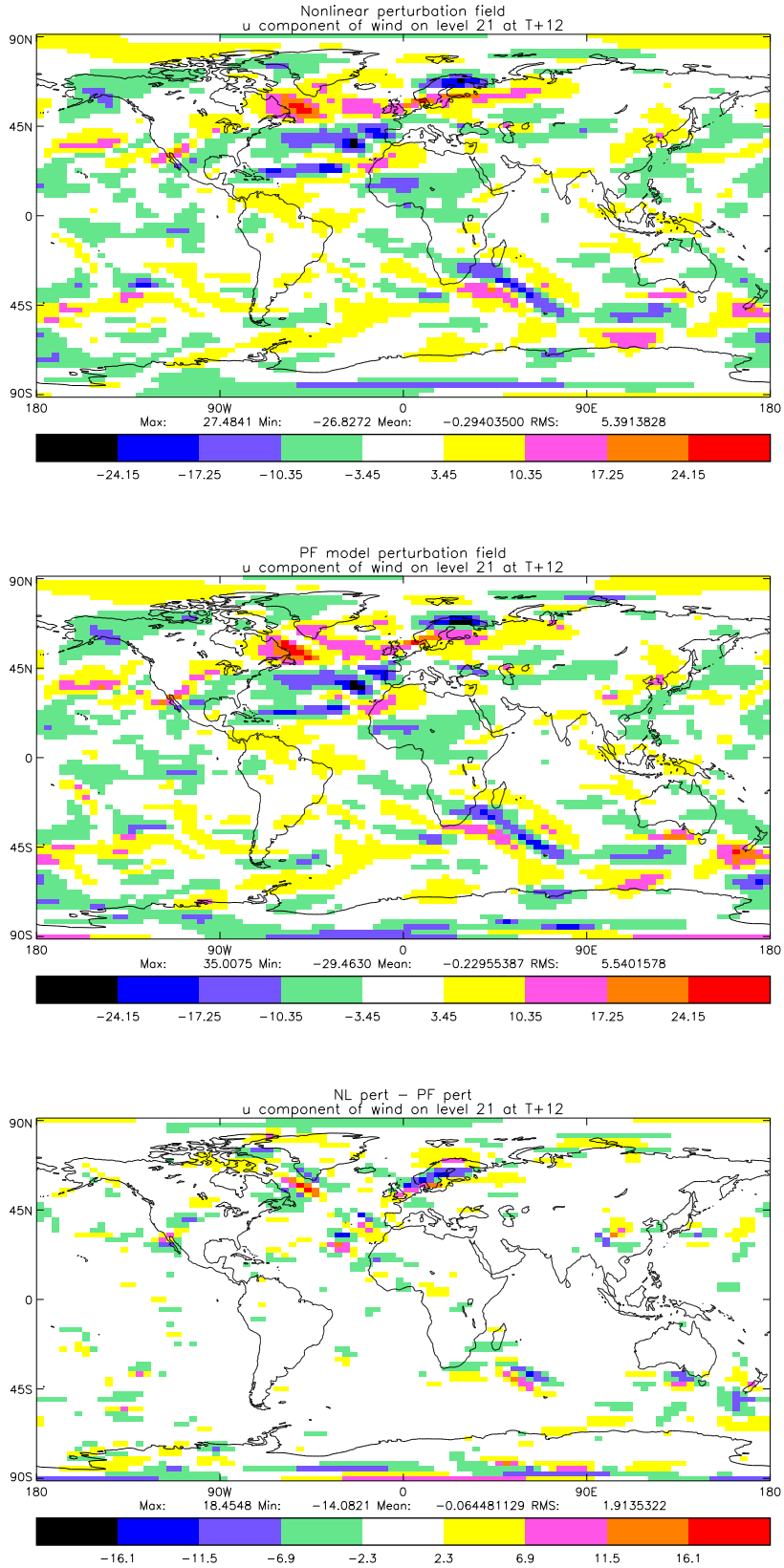

Figure 4: Zonal velocity perturbations from the Predictor-Corrector scheme

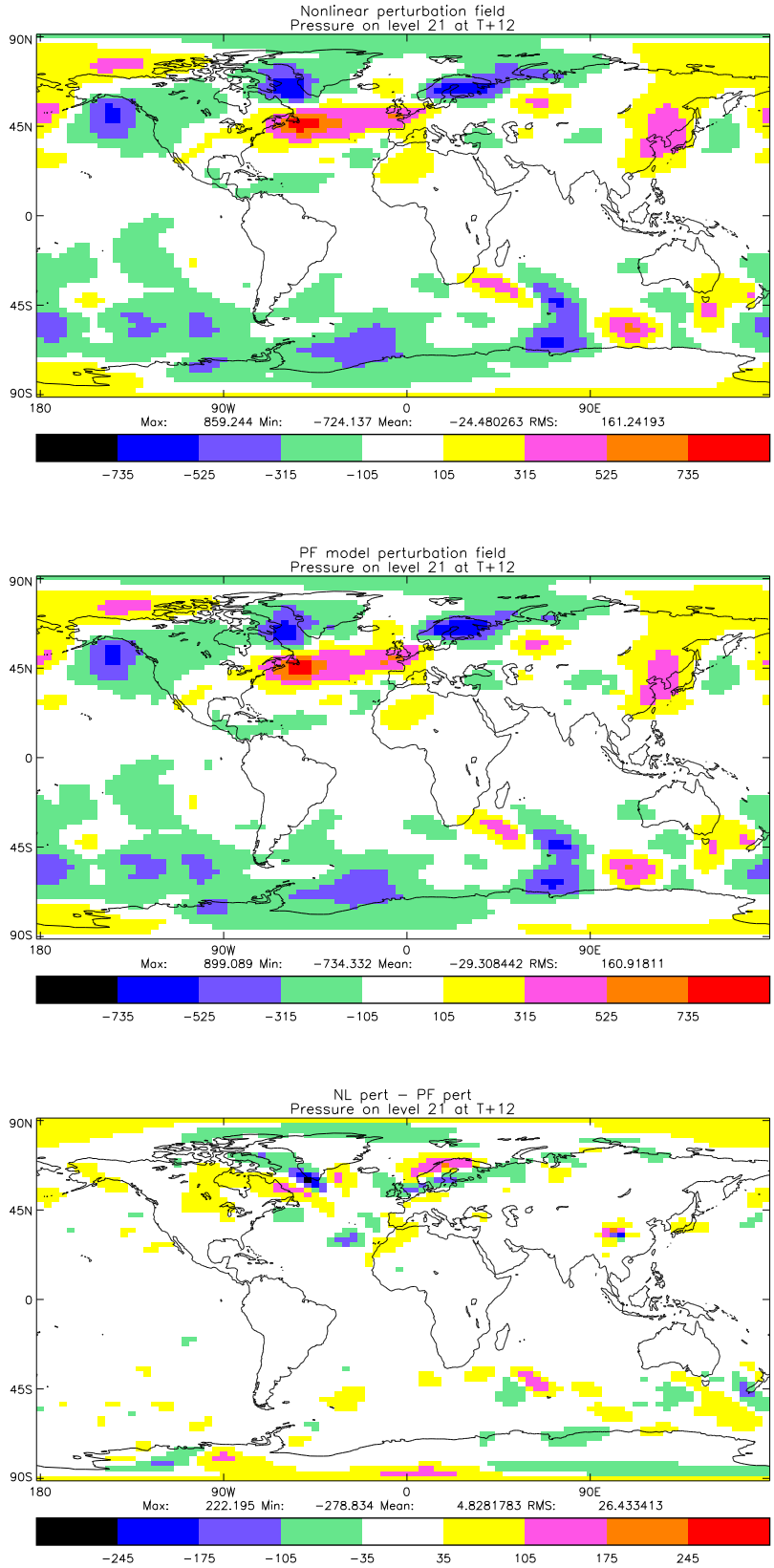

Figure 5: Pressure perturbations from the Predictor-Corrector scheme

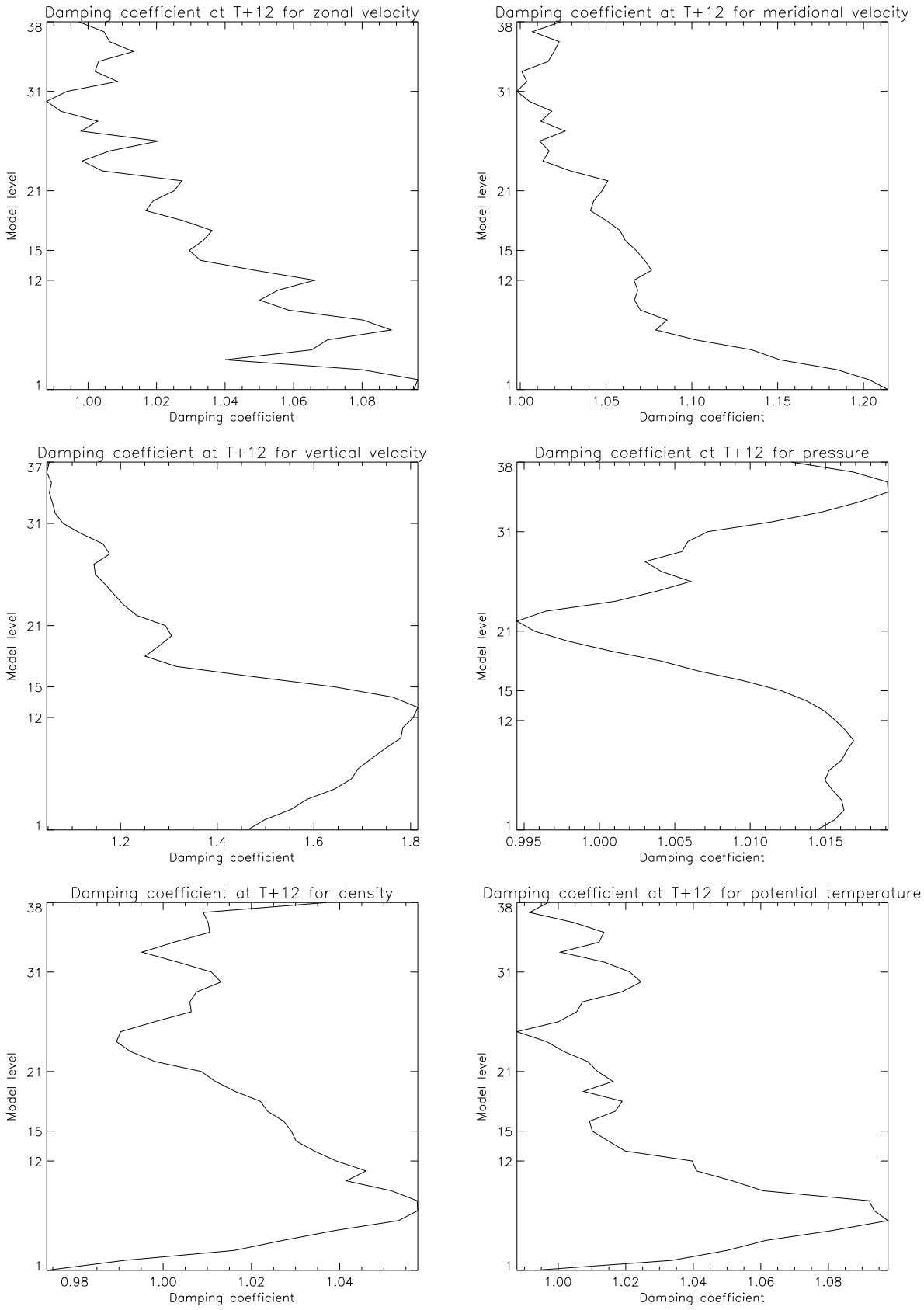

Figure 6: Damping coefficients from the Predictor-Corrector scheme

with the PF model). The results from this scheme can be seen in Figures 7 and 8, which are again on Level 21 after 12 hours. These results show excellent agreement with those obtained using the new predictor-corrector scheme. Thus the intended goal of producing a scheme which mimics the implicit scheme in the New Dynamics has been achieved with the new predictor-corrector scheme.

4 Stability of the Predictor-Corrector scheme

In order to assess the stability of the predictor-corrector scheme, two simplified cases will be considered. Firstly, the scheme will be applied to inertial oscillations, and then to the shallow water equations.

4.1 Inertial Oscillations

The equation to be considered is

$$\frac{D\mathbf{v}}{Dt} = -f\mathbf{k} \times \mathbf{v} \tag{13}$$

where f is the coriolis parameter and $\mathbf{k}$ is the unit vector in the upward normal direction. This describes inertial oscillations having frequency f and constant amplitude. Discretising (13) in a semi-implicit semi-Lagrangian manner yields

$$\mathbf{v}^{n+1} - \mathbf{v}_d^n = -f\Delta t\mathbf{k} \times [(1-\alpha)\,\mathbf{v}_d^n + \alpha\mathbf{v}^*] \tag{14}$$

where Δt is the computational timestep, n is the time level, d denotes departure point, unsubscripted velocities v are evaluated at the arrival point and α is the implicit weight - zero giving an explicit scheme, and unity an implicit scheme. In the new predictor-corrector scheme

$$\mathbf{v}^* = \mathbf{v}_d^n - f\Delta t\mathbf{k} \times [(1-\alpha)\,\mathbf{v}_d^n + \alpha\mathbf{v}^n]\,, \tag{15}$$

hence,

$$\mathbf{v}^{n+1} - \mathbf{v}_d^n = -f\Delta t\mathbf{k} \times [\mathbf{v}_d^n - \alpha f\Delta t\mathbf{k} \times [(1-\alpha)\,\mathbf{v}_d^n + \alpha\mathbf{v}^n]] \tag{16}$$

Assuming infinitesimal displacements (effectively a small amplitude approximation) gives,

$$\mathbf{v}^{n+1} - \mathbf{v}^n = -f\Delta t\mathbf{k} \times [\mathbf{v}^n - \alpha f\Delta t\mathbf{k} \times [(1-\alpha)\,\mathbf{v}^n + \alpha\mathbf{v}^n]]\,. \tag{17}$$

in which all velocities are evaluated at the arrival point. This can then be rearranged to give

$$\mathbf{v}^{n+1} = \left(1 - \alpha f^2\Delta t^2 - f\Delta t\mathbf{k}\times\right)\mathbf{v}^n \tag{18}$$

Thus the magnitude of vector $\mathbf{v}$ is amplified by a factor of

$$\left[1 + f^2\Delta t^2\left(1 - 2\alpha + \alpha^2 f^2\Delta t^2\right)\right]^{\frac{1}{2}}. \tag{19}$$

The requirement for stability is that this factor is less than unity; thus it is required that

$$1 - 2\alpha + \alpha^2 f^2\Delta t^2 \leq 0. \tag{20}$$

This is satisfied if

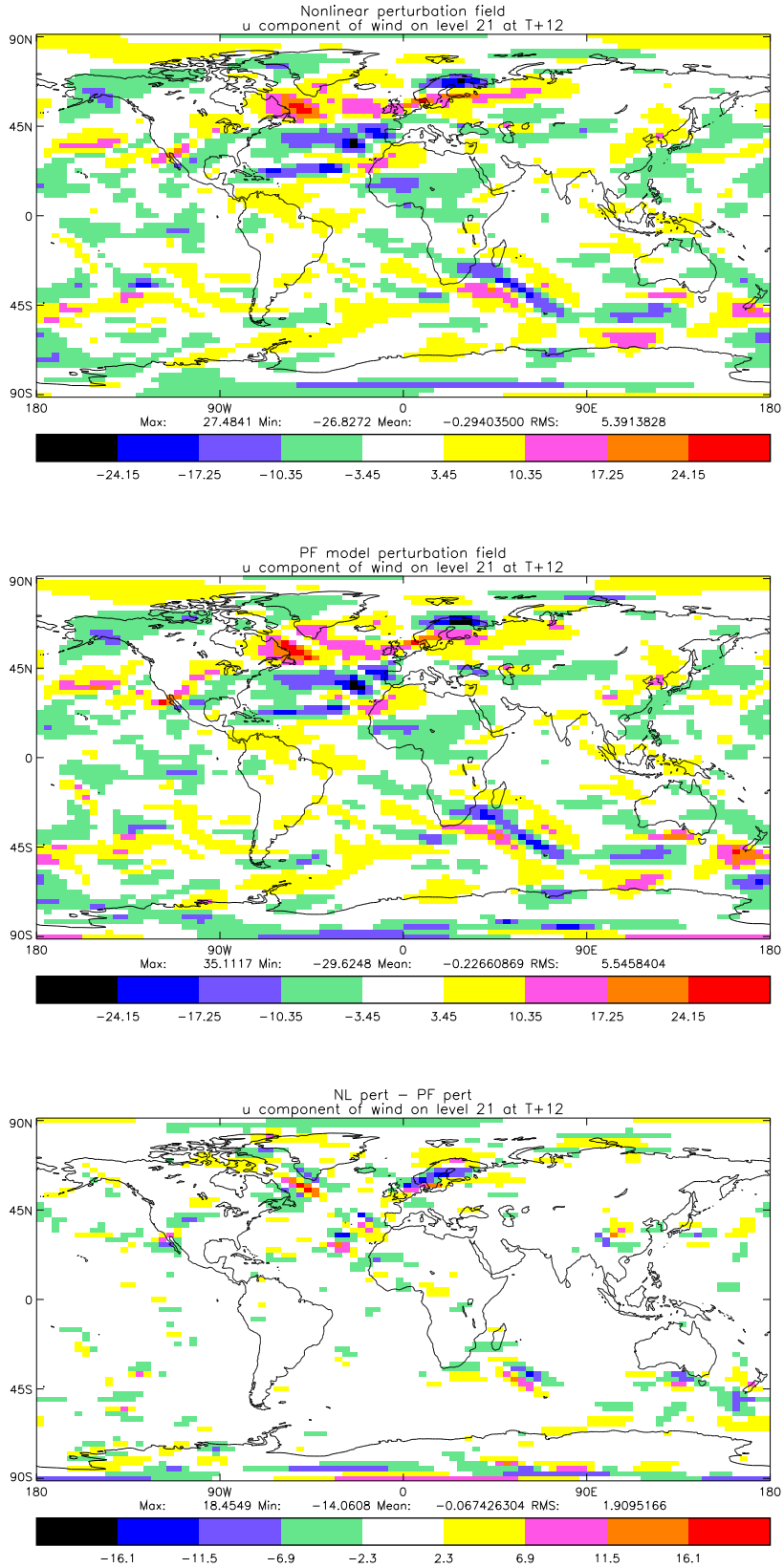

Figure 7: Zonal velocity perturbations from the New Dynamics scheme

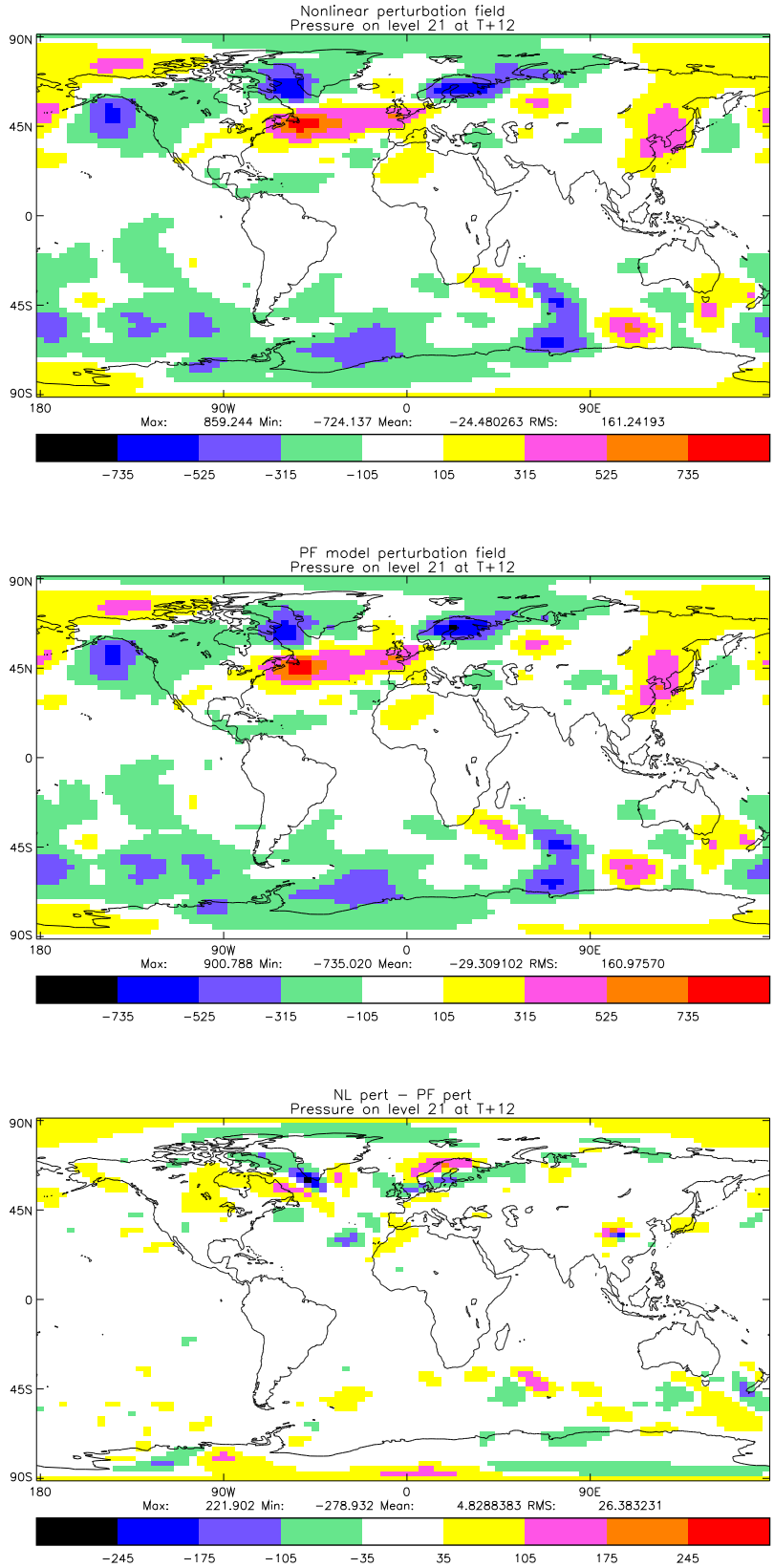

Figure 8: Pressure perturbations from the New Dynamics scheme

$$\frac{1-\sqrt{1-f^2\Delta t^2}}{f^2\Delta t^2} \leq \alpha \leq \frac{1+\sqrt{1-f^2\Delta t^2}}{f^2\Delta t^2} \tag{21}$$

which gives

$$0.51 \leq \alpha \leq 31.5 \tag{22}$$

if $f\Delta t = \frac{1}{4}$ which is approximately the maximum value used in practice. Equivalently, using the current value of $\alpha = 0.6$ the timestep required for stability is

$$\Delta t \leq 5100s, \tag{23}$$

where the maximum value used in the PF model is 1800s. Hence this scheme is stable in the case of inertial oscillations using the same semi-implicit weight as used in the New Dynamics.

4.2 Shallow Water Equations

In familiar notation, the shallow water equations are

$$\frac{Du}{Dt} = fv - g\frac{\partial h}{\partial x} \tag{24}$$

$$\frac{Dv}{Dt} = -fu - g\frac{\partial h}{\partial y} \tag{25}$$

$$\frac{Dh}{Dt} = -h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{26}$$

These are considered on the f-plane (f = constant) so Rossby waves will be stationary (relative to an assumed basic state of rest and uniform depth h). Semi-Lagrangian discretisation of (24)-(26) and small amplitude approximation (as in (17)) leads to

$$u^{n+1} - u^n = f\alpha\Delta t v^* + f(1-\alpha)\Delta t v^n - \frac{g}{2}\Delta t\frac{\partial}{\partial x}\left(h^{n+1} + h^n\right) \tag{27}$$

$$v^{n+1} - v^n = -f\alpha\Delta t u^* - f(1-\alpha)\Delta t u^n - \frac{g}{2}\Delta t\frac{\partial}{\partial y}\left(h^{n+1} + h^n\right) \tag{28}$$

$$h^{n+1} - h^n = -\Delta t\frac{h_0}{2}\left(\frac{\partial}{\partial x}\left[u^{n+1} + u^n\right] + \frac{\partial}{\partial y}\left[v^{n+1} + v^n\right]\right) \tag{29}$$

with the predictors u^*, v^* given by

$$u^* = u^n + f\Delta t v^n - g\Delta t\frac{\partial}{\partial x}(h^n) \tag{30}$$

$$v^* = v^n - f\Delta t u^n - g\Delta t\frac{\partial}{\partial y}(h^n) \tag{31}$$

The terms involving the height, h, are centrally differenced in time, which is known to be stable, and the coriolis terms are given the implicit weight α, which determines how much of the terms are evaluated at the old time level, and how much by the predictors for the next time level. Substitution of (30), (31) into (27), (28) gives

$$u^{n+1} = \left(1-\alpha f^2\Delta t^2\right)u^n + f\Delta t v^n - \alpha f g\Delta t\frac{\partial h^n}{\partial y} - \frac{g}{2}\Delta t\frac{\partial}{\partial x}\left(h^{n+1}+h^n\right) \quad (32)$$

$$v^{n+1} = \left(1-\alpha f^2\Delta t^2\right)v^n - f\Delta t u^n + \alpha f g\Delta t\frac{\partial h^n}{\partial x} - \frac{g}{2}\Delta t\frac{\partial}{\partial y}\left(h^{n+1}+h^n\right) \quad (33)$$

The equations (32), (33) and (29) can be rewritten in terms of divergence, D, and vorticity, ζ as

$$D^{n+1} = \left(1-\alpha\Delta t^2 f^2\right)D^n + f\Delta t\zeta^n - \frac{g}{2}\Delta t\left(\nabla^2 h^n + \nabla^2 h^{n+1}\right) \quad (34)$$

$$\zeta^{n+1} = \left(1-\alpha\Delta t^2 f^2\right)\zeta^n - f\Delta t D^n + \alpha f g\Delta t^2\nabla^2 h^n \quad (35)$$

$$h^{n+1} = h^n - \frac{1}{2}h_0\Delta t\left(D^{n+1}+D^n\right) \quad (36)$$

where

$$D^n = \frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y} \quad (37)$$

$$\zeta^n = \frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}. \quad (38)$$

for all time levels n. The behaviour of sinusoidal, wave-like solutions can be investigated by applying

$$(D,\zeta,h) = \left(\hat{D},\hat{\zeta},\hat{h}\right)exp\left[i\left(kx+ly-\omega t\right)\right] \quad (39)$$

and setting

$$E = e^{-i\omega\Delta t}. \quad (40)$$

This results in the following equations

$$\hat{D}\left(E-1+\alpha\Delta t^2 f^2\right) + \hat{\zeta}\left(-f\Delta t\right) + \hat{h}\left(-\frac{1}{2}g\Delta t\left(k^2+l^2\right)(1+E)\right) = 0 \quad (41)$$

$$\hat{\zeta}\left(E-1+\alpha\Delta t^2 f^2\right) + \hat{D}\left(f\Delta t\right) + \hat{h}\left(\alpha f g\Delta t^2\left(k^2+l^2\right)\right) = 0 \quad (42)$$

$$\hat{h}\left(E-1\right) + \hat{D}\left(\frac{1}{2}h_0\Delta t\left(1+E\right)\right) = 0 \quad (43)$$

which can be written in matrix form as

$$\mathbf{M}\begin{pmatrix}\hat{D}\\ \hat{\zeta}\\ \hat{h}\end{pmatrix} = \begin{pmatrix} E-1+\alpha\Delta t^2 f^2 & -f\Delta t & -\frac{g}{2}\Delta t\left(k^2+l^2\right)(1+E) \\ f\Delta t & E-1+\alpha\Delta t^2 f^2 & \alpha f g\Delta t^2\left(k^2+l^2\right) \\ \frac{1}{2}h_0\Delta t\left(1+E\right) & 0 & E-1 \end{pmatrix}\begin{pmatrix}\hat{D}\\ \hat{\zeta}\\ \hat{h}\end{pmatrix} = 0. \quad (44)$$

For non-trivial solutions, the determinant of $\mathbf{M}$ must be zero, giving a cubic equation for E. Substituting the quantities

$$A = f^2 \Delta t^2 \quad (45)$$

$$B = \frac{1}{4} g \Delta t^2 \left(k^2 + l^2\right) h_0 \quad (46)$$

this cubic equation becomes

$$(E - 1 + \alpha A)\left[(E - 1 + \alpha A)(E - 1) + B(1 + E)^2\right] + A(E - 1) - 2B\alpha A(1 + E) = 0. \quad (47)$$

Direct substitution shows that $E = 1$ is a solution of (47) for all values of α. This solution is the geostrophic (Rossby) mode, which is still stationary using this scheme - a desirable property. Factorising out $E - 1$ leaves the two solutions for the rotation-modified surface gravity waves. They are solutions to the quadratic equation

$$E^2(1 + B) + E(2\alpha A + 2B + \alpha AB - 2) + \left(\alpha^2 A^2 + B + A - 2\alpha A + \alpha AB + 1\right) = 0, \quad (48)$$

and thus satisfy

$$E(2 + 2B) = 2 - 2\alpha A - 2B - \alpha AB \pm \sqrt{\alpha^2 A^2 B^2 - 16B - 4A - 4AB + 8\alpha AB}. \quad (49)$$

With A and B both being positive quantities with typical values significantly less than unity, the discriminant of the above equation is typically negative. For this case,

$$|E|^2 = 1 + \frac{A}{1 + B}\left(1 + \alpha^2 A + (B - 2)\alpha\right) \quad (50)$$

Considering $\alpha = \frac{1}{2}$, gives

$$|E|^2 = 1 + \frac{A}{2(1 + B)}\left(\frac{A}{2} + B\right) \quad (51)$$

which, recalling the definitions (45) and (46) of A and B give an $O(\Delta t^4)$ instability, which is very weak, and almost certain to be acceptable over the short ranges required by the PF model. Moreover, the value of α as used in the New Dynamics is 0.6, and this gives

$$|E|^2 = 1 + \frac{A}{25(1 + B)}(9A + 15B - 5) \quad (52)$$

which is less than unity given A and B have sufficiently small magnitudes; almost certainly the case. Hence, as in the case of inertial oscillations, it appears that an implicit weight of one-half is not sufficient for stability, but values not much greater produce a stable scheme.

5 Conclusions

The new predictor-corrector scheme for the coriolis terms has been shown to produce results very similar to those given by a New Dynamics style scheme, whilst having a simple adjoint. The scheme has been shown to remove the damping produced by the previously implemented scheme, and to produce much more accurate results. Stability analyses of the new scheme have also been carried out, showing the scheme to be stable in the cases of inertial oscillations and the linearised shallow water equations given a large enough implicit weight. The implicit weight used in the coriolis terms in the New Dynamics, 0.6, has been shown to be sufficient in this new scheme. The scheme has been successfully implemented in the PF and adjoint models using an implicit weight of 0.6.

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[1] Staniforth, A. et. al., UM5.2 Model Formulation, Unified Model Documentation Paper 15, Met Office, 2001.

[2] Gadd A.J., Two refinements of the split explicit integration scheme, *Quart. J. Roy. Meteor. Soc.*, 106, 215-220, 1980.